SOME PROPERTIES OF $\alpha$-HARMONIC MEASURE

DIMITRIOS BETSAKOS

Abstract. The $\alpha$-harmonic measure is the hitting distribution of symmetric $\alpha$-stable processes upon exiting an open set in $\mathbb{R}^n$ ($0 < \alpha < 2$, $n \geq 2$). It can also be defined in the context of Riesz potential theory and the fractional Laplacian. We prove some geometric estimates for $\alpha$-harmonic measure.

1. Introduction

In the 1930’s, O.Frostman and M.Riesz developed a potential theory on $\mathbb{R}^n$, $n \geq 2$, based on the Riesz kernel

$$k_\alpha(x) = \frac{A(n, \alpha)}{|x|^{n-\alpha}}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $0 < \alpha < 2$ and $A(n, \alpha)$ is a constant. When $\alpha = 2$, the Riesz kernel coincides with the kernel of the classical potential theory, the Newtonian kernel ($n \geq 3$).

The $\alpha$-harmonic functions are defined by a mean value property (involving the parameter $\alpha$), analogous to the classical one. Equivalently, they are the solutions of the equation $\Delta^{\alpha/2} u = 0$, where $\Delta^{\alpha/2}$ is the fractional Laplacian, a non-local integro-differential operator.

A function $u : \mathbb{R}^n \to \mathbb{R}$ which is $\alpha$-harmonic in an open set $D$ is determined by its exterior values (its values in $D^c := \mathbb{R}^n \setminus D$). If $B$ is a Borel set in $D^c$, the $\alpha$-harmonic measure of $B$ with respect to $D$ is the $\alpha$-harmonic function $u$ in $D$ with exterior values $u = \chi_B$ on $D^c$. The $\alpha$-harmonic measure of $B$ with respect to $D$, evaluated at the point $x \in \mathbb{R}^n$, will be denoted by $\omega^\alpha_D(x, B)$. For fixed $x \in D$, $\omega^\alpha_D(x, \cdot)$ is a Borel probability measure on $D^c$.

Both classical and $\alpha$-harmonic measures have symmetry properties and satisfy the Carleman principle (domain monotonicity) and the Harnack principle. The latter implies that if $\omega^\alpha_D(x, B) = 0$ for some $x \in D$, then $\omega^\alpha_D(y, B) = 0$ for all $y \in D$; we say then that $B$ is a $D$-null set.

The classical harmonic measure is defined (as function) in a domain $D$ and is supported (as measure) on the boundary of $D$. The $\alpha$-harmonic measure is defined (as function) in whole $\mathbb{R}^n$ and is supported (as measure) in the exterior of $D$. These properties become transparent when are viewed from the probabilistic point of view. The classical harmonic measure is the hitting distribution of Brownian motion upon exiting $D$, while the $\alpha$-harmonic measure is the hitting distribution of symmetric $\alpha$-stable process. This is a Hunt process with discontinuous paths. Thus its paths may jump from one component of $D$ to another and may hit $D^c$ (upon exiting $D$) at points of $(\partial D)^c$ and not necessarily at points of $\partial D$. 

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In Section 3, we prove some geometric estimates for \( \alpha \)-harmonic measure involving symmetric or polarized open sets \( D \). Although the corresponding inequalities for classical harmonic measure are almost trivial, we will see that the proofs for \( \alpha \)-harmonic measure are not simple. Theorems 1 and 2 were proved in [2] under more restrictive conditions; (in [2, Theorem 3], the open set \( D \) is assumed to be bounded with boundary satisfying an exterior cone condition).

2. Background

2.1. \( \alpha \)-harmonic functions. The M. Riesz kernels in \( \mathbb{R}^n \), \( n \geq 2 \), are the functions

\[
 k_{\alpha}(x) = \frac{A(n, \alpha)}{|x|^{n-\alpha}}, \quad x \in \mathbb{R}^n \setminus \{0\},
\]

where \( 0 < \alpha < n \) and

\[
 A(n, \gamma) = \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)2^{\gamma} \pi^{n/2}}, \quad -n < \gamma < n, \quad \gamma \neq 0, -2, -4, \ldots.
\]

These kernels include as special and limiting cases the kernels of the classical potential theory: the Newtonian kernel \( (n \geq 3, \alpha = 2) \) and the logarithmic kernel \( (n = 2, \alpha \to 2) \); see [12, Ch.I]. From now on, we assume that \( 0 < \alpha < 2 \). We denote the \( n \)-dimensional Lebesgue measure by \( m_n \).

**Definition 1.** Let \( D \) be an open set in \( \mathbb{R}^n \), \( n \geq 2 \). A function \( u : \mathbb{R}^n \to \mathbb{R} \) is called \( \alpha \)-harmonic in \( D \) if

(a) \( u \) is continuous in \( D \);
(b) \( u \) is in \( \mathcal{L}^1 \); that is, \( u \) is locally integrable on \( \mathbb{R}^n \) and

\[
 \int_{|x| > 1} \frac{|u(x)|}{|x|^{n+\alpha}} m_n(dx) < \infty;
\]

(c) for every ball \( B(x_0, r) \) with closure in \( D \),

\[
 u(x_0) = \int_{\mathbb{R}^n} u(x) e_{\alpha}^r(x - x_0) \, m_n(dx),
\]
where
\[ \varepsilon^{(r)}(x) = \begin{cases} \frac{\Gamma(n/2) \sin(\pi \alpha/2)}{\pi^{n/2} \Gamma(n/2 + 1)} \frac{r^n}{|x|^n}, & |x| > r, \\ 0, & |x| < r. \end{cases} \]

**Definition 2.** Let \( f \in \mathcal{L}^1 \). For \( \varepsilon > 0 \) and \( x \in \mathbb{R}^n \), we define
\[ \Delta^{\alpha/2} f(x) = \mathcal{A}(n, -\alpha) \int_{|y-x|>\varepsilon} \frac{f(y) - f(x)}{|y-x|^{n+\alpha}} m_n(dy) \]
and
\[ \Delta^{\alpha/2} f(x) = \lim_{\varepsilon \downarrow 0} \Delta^{\alpha/2} f(x), \]
whenever the limit exists.

By [6, Theorem 3.9], a function \( u \) defined on \( \mathbb{R}^n \) is \( \alpha \)-harmonic in an open set \( D \) if and only if it is continuous in \( D \) and \( \Delta^{\alpha/2} u = 0 \) in \( D \).

### 2.2. The Dirichlet problem for \( \alpha \)-harmonic functions. (See [12, Ch.IV], [3, Ch.VII], [15]). The Perron-Wiener-Brelot method can be applied for the solution of the Dirichlet problem for \( \alpha \)-harmonic functions. Let \( D \) be an open set in \( \mathbb{R}^n \).

An \( \alpha \)-subharmonic function in \( D \) is an \( \mathcal{L}^1 \) function which is upper semicontinuous in \( D \) and satisfies the inequality
\[ u(x_o) \leq \int_{\mathbb{R}^n} u(x) \varepsilon^{(r)}(x-x_o) m_n(dx), \]
for every ball \( B(x_o, r) \) with closure in \( D \).

Let \( C(D^c) \) be the class of functions \( f \) continuous in \( D^c \) satisfying
\[ \int_{D^c \cap \{|x|>1\}} \frac{|f(x)|}{|x|^{n+\alpha}} m_n(dx) < \infty, \]
and \( H(D) \) be the class of functions on \( \mathbb{R}^n \), \( \alpha \)-harmonic in \( D \). The lower Perron family of a function \( f \in C(D^c) \) is the family \( \mathcal{P}_f \) of all functions \( u \) which are \( \alpha \)-subharmonic in \( D \) and satisfy the inequalities \( u \leq f \) in \( (D)^c \) and
\[ \limsup_{D^c \ni x \to \zeta} u(x) \leq f(\zeta), \quad \forall \zeta \in \partial D. \]

Define
\[ H_f(x) := \sup\{u(x) : u \in \mathcal{P}_f\}, \quad x \in \mathbb{R}^n. \]

Then \( H_f \) is \( \alpha \)-harmonic in \( D \). The definition of regular and irregular boundary points and their characterization by Wiener’s criterion are similar to their classical analogs. The function \( H_f \) has limit \( f(\zeta) \) at each regular boundary point \( \zeta \). We say that \( H_f \) is the Perron solution of the Dirichlet problem in \( D \) with exterior values \( f \).

The operator \( f \mapsto H_f \) is a positive linear operator from \( C(D^c) \) into \( H(D) \). Hence for each \( x \in \mathbb{R}^n \), there is a measure \( \omega^D_n(x, \cdot) \) on \( D^c \) such that
\[ H_f(x) = \int_{D^c} f(y) \omega^D_n(x, dy), \quad x \in \mathbb{R}^n. \]
This measure is the \( \alpha \)-harmonic measure for \( D \) evaluated at \( x \).
In a similar manner, one can define the upper and the lower Perron family for any Borel function on $D^c$ and consider the corresponding generalized solution for the Dirichlet problem; see [3] for more details.

2.3. **Symmetric stable processes.** (See [4], [5], [6], [10], [11], [14], [3], [8]). The fractional Laplacian $\Delta^{\alpha/2}$ is the characteristic operator of the symmetric $\alpha$-stable process $\{X_t, t \in [0, \infty)\}$ in $\mathbb{R}^n$. This is a Lévy process (homogeneous and with independent increments) with transition density $p_t(x, y) = p_t(y, x) = p_t(x - y)$ (relative to the Lebesgue measure) uniquely determined by its Fourier transform

$$e^{ix\xi} p_t(x) m_n(dx) = e^{-t|\xi|^\alpha}. \quad (2.9)$$

When $\alpha = 2$, we get a Brownian motion running at twice the speed. The probability measures and the corresponding expectations of the process $\{X_t\}$ starting at $x \in \mathbb{R}^n$ will be denoted by $P^x$ and $E^x$.

The symmetric $\alpha$-stable process $\{X_t\}$ is a strong Feller and a Hunt process. For $A \subset \mathbb{R}^n$, we put

$$T^A = \inf\{t > 0 : X_t \notin A\}, \quad (2.10)$$

the first exit time from $A$. A Borel function $u$ defined on $\mathbb{R}^n$ is $\alpha$-harmonic in an open set $D \subset \mathbb{R}^n$ if and only if

$$u(x) = E^x u(X_{T^D}), \quad x \in U, \quad (2.11)$$

for every bounded open set $U$ with closure $\overline{U}$ contained in $D$. If $D \subset \mathbb{R}^n$ is open and $B$ is a Borel subset of $D^c$, then

$$\omega^D_\alpha(x, B) = P^x(X_{T^D} \in B), \quad x \in \mathbb{R}^n. \quad (2.12)$$

2.4. **Riesz capacity.** (See [12, Chapter II]). If $K$ is a compact set in $\mathbb{R}^n$ and $\mu$ is a probability Borel measure on $K$, the $\alpha$-energy of $\mu$ is

$$I_\alpha(\mu) = \int_K \int_K k_\alpha(x - y) \mu(dx) \mu(dy). \quad (2.13)$$

The $\alpha$-capacity of $K$ is defined by

$$C_\alpha(K) = \left( \inf_{\mu} I_\alpha(\mu) \right)^{-1}, \quad (2.14)$$

where the infimum is taken over all probability Borel measures on $K$.

For a Borel set $E \subset \mathbb{R}^n$, we define

$$C_\alpha(E) = \sup\{C_\alpha(K) : K \subset E \text{ compact}\}. \quad (2.15)$$

By Choquet capacitability theorem [12, Theorem 2.8, p.156],

$$C_\alpha(E) = \inf\{C_\alpha(G) : E \subset G \text{ open}\}. \quad (2.16)$$

The $\alpha$-capacity is a geometric quantity because of its expression as transfinite diameter; see [12, Ch.II, §3]. It can also be characterized in terms of symmetric stable processes; see references in [2].
2.5. Null sets. There is no known geometric characterization of null sets for \( \alpha \)-harmonic measure. If a boundary set has zero \( \alpha \)-capacity, then it has also zero \( \alpha \)-harmonic measure; see [12]. The following lemmas provide more refined necessary or sufficient conditions.

**Lemma 1.** [15, Theorem 1'] Let \( D \) be an open set in \( \mathbb{R}^n \) and \( F \) be a subset of \( \partial D \) with \( m_{\alpha}(F) = 0 \). Suppose that there exists \( c > 0 \) such that for all \( x \in D \),
\[
m_{\alpha}(D^c \cap B(x, 2d(x, F))) > c d(x, F)^n.
\]
Then \( F \) is \( D \)-null.

**Lemma 2.** [15, Theorem 3] Let \( D \) be an open set in \( \mathbb{R}^n \) and \( F \) be a subset of \( \partial D \) with \( C_{\alpha}(F) > 0 \). If
\[
\lim_{r \to 0} C_{\alpha}(\{x \in D^c : 0 < d(x, F) \leq r\}) = 0,
\]
then \( F \) is not \( D \)-null.

**Lemma 3.** Suppose that \( D \) and \( \Omega \) are open sets in \( \mathbb{R}^n \) with \( D \subset \Omega \). Let \( A = \Omega \setminus D \) and assume that \( A \) is \( D \)-null. Then \( C_{\alpha}(A) = 0 \).

*Proof.* By Choquet capacitability theorem [12, Theorem 2.8, p.156], \( A \) is capacitatable. Assume first that \( A \) is compact. Then \( d(A, \partial \Omega) > 0 \). For \( 0 < r < d(A, \partial \Omega) \), the set
\[
\{x \in D^c : 0 < d(x, A) \leq r\}
\]
is empty. By Lemma 2, \( C_{\alpha}(A) = 0 \).

Next assume that \( A \) is bounded. Let
\[
A_k = \left\{ x \in A : d(x, \partial \Omega) \geq \frac{1}{k} \right\}, \quad k \in \mathbb{N}.
\]
Then \( A_k \) is compact. Hence \( C_{\alpha}(A_k) = 0 \) for all \( k \). By the subadditivity of \( \alpha \)-capacity, \( C_{\alpha}(A) = 0 \). Finally, for unbounded \( A \) we consider the sequence of bounded sets
\[
A_m = \{x \in A : |x| \leq m\}, \quad m \in \mathbb{N}
\]
and conclude as above that \( C_{\alpha}(A) = 0 \). \( \Box \)

2.6. The minimum principle in Riesz potential theory. We will need some extensions of the minimum principle for \( \alpha \)-superharmonic functions; see [12, pp. 115, 183].

**Lemma 4.** Let \( D \) be an open set in \( \mathbb{R}^n \) and \( u : \mathbb{R}^n \to (-\infty, +\infty] \) be a function which is \( \alpha \)-superharmonic in \( D \) and lower semicontinuous on \( \overline{D} \). Suppose that there exists a constant \( M \in \mathbb{R} \) such that \( u \geq M \) in \( D^c \). Then \( u \geq M \) in \( \mathbb{R}^n \). If \( u(x_o) = M \) for some \( x_o \in D \), then \( u = M \) in \( \mathbb{R}^n \).
Define \( v(x) = u(x) - M, \ x \in \mathbb{R}^n \). Then \( v \) is lower semicontinuous on \( \partial D \).

Also, for \( \zeta \in \partial D \),

\[
\liminf_{\partial D \ni x \to \zeta} v(x) = \liminf_{\partial D \ni x \to \zeta} u(x) - M \geq u(\zeta) - M \geq 0.
\]

Suppose that there exists a point \( x_0 \in D \) such that

\[
\min_{\partial D} v = v(x_0) < 0.
\]

Take \( r > 0 \) sufficiently small so that the ball of radius \( r \), centered at \( x_o \), lies in \( D \). Then \( v(x_0) < \varepsilon_{\alpha,x_0}^{(r)} \); indeed, if \( v(x_0) = \varepsilon_{\alpha,x_0}^{(r)} \), then \( v = v(x_0) < 0 \) a.e. in \( \{|x-x_o| > r\} \), and therefore

\[
\liminf_{x \to \zeta \in D} v(x) \leq v(x_0) < 0,
\]

contradicting (2.17). Hence

\[
v(x_0) < \varepsilon_{\alpha,x_0}^{(r)} \varepsilon \varepsilon_{\alpha,x_0}^{(r)} v = \varepsilon_{\alpha,x_0}^{(r)} u - M \leq u(x_0) - M = v(x_0),
\]

which is absurd. We conclude that the minimum of \( v \) on \( \partial D \) is non-negative and therefore \( v(x) \geq M \) for all \( x \in \mathbb{R}^n \).

If \( u(x_o) = M \) for some \( x_o \in D \), then for all sufficiently small \( r > 0 \),

\[
0 = v(x_0) \geq \varepsilon_{\alpha,x_0}^{(r)} v.
\]

This implies \( v = 0 \) a.e. in \( \mathbb{R}^n \); that is, \( u = M \) a.e. in \( \mathbb{R}^n \). If \( x \in D \), then [12, p.114]

\[
u(x) = \lim_{r \to 0} \varepsilon_{\alpha,x_0}^{(r)} u = M.
\]

Hence \( u = M \) in \( D \).

**Lemma 5.** Let \( D \) be an open set in \( \mathbb{R}^n \) and \( u : \mathbb{R}^n \to (-\infty, +\infty] \) be a function \( \alpha \)-superharmonic in \( D \). Assume that

(i) \( u \) is bounded below in \( D \);

(ii) \( u \) is lower semicontinuous in \( \partial D \setminus E \), where \( E \) is a subset of \( \partial D \) with \( \infty \notin E \) and \( C_\alpha(E) = 0 \); (of course, if \( E \subseteq \mathbb{R}^n \) then \( \infty \notin E \));

(iii) \( \liminf_{\partial D \ni x \to \zeta} u(x) \geq M \), for some \( M \in \mathbb{R} \) and all \( \zeta \in \partial D \setminus E \);

(iv) \( u(x) \geq M \), for all \( x \in (\partial D)^c \).

Then \( u(x) \geq M \), for all \( x \in D \). Moreover, if \( u(x_o) = M \) for some \( x_o \in D \), then \( u = M \) in \( D \).

**Proof.** For \( n \in \mathbb{N} \), let \( A_n \) be an open set such that \( E \subseteq A_n \) and \( C_\alpha(A_n) \leq \frac{1}{n} \). Then the set \( E_1 := \bigcap_{n=1}^\infty A_n \) is a \( G_\delta \)-set such that \( E \subseteq E_1 \) and \( C_\alpha(E_1) = 0 \).

There exists a measure \( \lambda \) on \( \mathbb{R}^n \) such that the Riesz potential \( U_\lambda^\alpha \) of \( \lambda \) has the following properties (see [12, p. 179]):

\[
U_\lambda^\alpha(x) = \infty, \ \forall x \in E_1 \cap \partial D \text{ and } U_\lambda^\alpha(x) < \infty, \ \forall x \notin E_1 \cap \partial D.
\]

For \( \varepsilon > 0 \), define

\[
u_1(x) = u(x) + \varepsilon U_\lambda^\alpha(x), \ x \in \mathbb{R}^n.
\]

The function \( u_1 \) is \( \alpha \)-superharmonic in \( D \). Moreover,

\[
\liminf_{\partial D \ni x \to \zeta} u_1(x) \geq M, \ \forall \zeta \in \partial D
\]
because $U_\alpha^l(x) \geq 0$, $\forall x \in \mathbb{R}^n$ and $U_\alpha^l(x) = \infty$, $\forall x \in E_1 \cap \partial D$. Also, since $U_\alpha^l$ is lower semicontinuous in $\mathbb{R}^n$ and

$$
\liminf_{x=\infty, \zeta \in E} [u(x) + \varepsilon U_\alpha^l(x)] = +\infty = u(\zeta) + \varepsilon U_\alpha^l(\zeta),
$$

$u_1$ is lower semicontinuous in $\overline{D}$.

We apply Lemma 4 to the function $u_1$ and conclude

$$u_1(x) = u(x) + \varepsilon U_\alpha^l(x) \geq M, \forall x \in D.$$

Since $\varepsilon > 0$ is arbitrary and $U_\alpha^l < \infty$ in $D$, it follows that $u \geq M$ in $D$.

Suppose next that $u(x_0) = M$ for some $x_0 \in D$. By the $\alpha$-mean value inequality $M = u(x_0) \geq \varepsilon U_\alpha^l(x_0)$, for all sufficiently small $r > 0$. It follows that $u = M$ a.e. in $\mathbb{R}^n$. If $x \in D$, then [12, p.114]

$$u(x) = \lim_{r \to 0} \varepsilon U_\alpha^l(x) = M.$$

Hence $u = M$ in $D$.

3. SOME GEOMETRIC PROPERTIES OF $\alpha$-HARMONIC MEASURE

Let $\Pi = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$. For $E \subset \mathbb{R}^n$, we denote by $\widehat{E}$ the reflection of $E$ in the $(n-1)$-dimensional plane $\Pi$. Thus we have

$$\widehat{E} = \{(x_1, \ldots, x_{n-1}, x_n) : (x_1, \ldots, x_{n-1}, -x_n) \in E\}.$$

We will also use the following notation: if $x = (x_1, \ldots, x_{n-1}, x_n)$ then $\hat{x} := (x_1, \ldots, x_{n-1}, -x_n)$; $E_+ := \{(x_1, \ldots, x_{n-1}, x_n) \in E : x_n > 0\}$; $E_\alpha := E \cap \Pi$; $E_- := \{(x_1, \ldots, x_{n-1}, x_n) \in E : x_n < 0\}$.

Let $E$ be any set in $\mathbb{R}^n$. We divide $E$ into three subsets $S, U, V$:

$$
\begin{align*}
S &= S_E = \{x \in E : \hat{x} \in E\} = E \cap \widehat{E}, \\
U &= U_E = \{x \in E : x \in E_+, \hat{x} \notin E\} = E_+ \setminus S_E, \\
V &= V_E = \{x \in E : x \in E_-, \hat{x} \notin E\} = E_- \setminus S_E.
\end{align*}
$$

$S$ is the symmetric part of $E$, $U$ is the upper non-symmetric part of $E$, and $V$ is the lower non-symmetric part of $E$. The sets $S, U, V$ are disjoint and $E = S \cup U \cup V$. Note that if $E$ is open, then its symmetric part $S$ is always open, while the sets $U, V$ are not necessarily open. We say that $E$ is symmetric with respect to $\Pi$ if $U = V = \emptyset$ and hence $E = S$. We say that $E$ is polarized with respect to $\Pi$ if $V = \emptyset$ and hence $E = S \cup U$.

Theorem 1. Let $S$ be an open set in $\mathbb{R}^n$. Suppose that $S$ is symmetric with respect to $\Pi$. Let $B \subset \mathbb{R}^n \cap S^c$ be a Borel set. Then

(i) $\omega^S_\alpha(x, B) \geq \omega^S_\alpha(\hat{x}, B)$, $x \in \mathbb{R}^n$,

(ii) $\omega^S_\alpha(x, B) \geq \omega^S_\alpha(x, \hat{B})$, $x \in \mathbb{R}^n$.

Figure 1: An illustration for Theorem 1.
Proof. For $x \in \mathbb{R}^n_+ \setminus S_+$, the inequalities (i) and (ii) are trivial. So we prove them for $x = s \in S_+$. Because of symmetry, the inequalities (i) and (ii) are equivalent. So we prove only the first one. By the inner regularity of $\alpha$-harmonic measure, we may and do assume that $B$ is a compact set in $\mathbb{R}^n_+ \cap S^c$. Take a decreasing sequence of compactly supported continuous functions $f_k : S^c \to [0,1]$ with $\text{supp} f_k \downarrow B$, $f_k \downarrow \chi_B$ and $f_k = 0$ in $(S^c)_-$. Then for the sequence of functions
\[H_{f_k}(x) := \int_{S^c} f_k(y) \omega^S_{\alpha}(x,dy), \ x \in \mathbb{R}^n,\]
we have $H_{f_k}(s) \downarrow \omega^S_{\alpha}(s,B)$, $s \in S$. Therefore it suffices to prove that
\[H_{f_k}(s) \geq H_k(s), \ s \in S_+, \ k \in \mathbb{N}. \tag{3.1}\]

Let $E$ be the set of irregular points of $\partial S$. By a classical result (see e.g. [12, p.296]), $C_\alpha(E) = 0$. There exists a $G_\delta$-set $E_1 \supset E$ with $C_\alpha(E_1) = 0$ and a measure $\lambda$ on $\mathbb{R}^n$ such that (see [12, p. 179]):
\[U_\lambda(x) = \infty, \ \forall x \in E_1 \cap \partial D \quad \text{and} \quad U_\lambda(x) < \infty, \ \forall x \in \mathbb{R}^n \setminus (E_1 \cup \partial D).\]

Because of symmetry, we may also assume that $U_\lambda(x) = U_\lambda(\tilde{x})$. Fix $k \in \mathbb{N}$ and $\varepsilon > 0$ and define
\[v(x) = H_{f_k}(x) - H_k(x) + \varepsilon U_\lambda(x), \ x \in \mathbb{R}^n. \tag{3.2}\]

We look at the boundary values of $v$ in $S_+$. Let $\zeta \in \partial(S_+)$. Case 1: $\zeta \in S_-$.

Then
\[\liminf_{S_+ \ni s \to \zeta} v(s) = \liminf_{S_+ \ni s \to \zeta} \varepsilon U_\lambda(s) \geq 0.\]

Case 2: $\zeta \in \partial(S_+) \setminus (S_- \cup E_1)$.

Then
\[\liminf_{S_+ \ni s \to \zeta} v(s) = f_k(\zeta) - 0 + \liminf_{S_+ \ni s \to \zeta} \varepsilon U_\lambda(s) \geq 0.\]

Case 3: $\zeta \in E_1$.

Then by the lower semicontinuity of $U_\lambda$,
\[\liminf_{S_+ \ni s \to \zeta} v(s) = \varepsilon U_\lambda(\zeta) = \infty.\]

Case 4: $S$ is unbounded and $\zeta = \infty$.

Let $B_1$ be the support of $f_k$. For $s \in S$, we have
\[H_{f_k}(s) = \int_{S^c} f_k(y) \omega^S_{\alpha}(s,dy) \leq \int_{B_1} \omega^S_{\alpha}(s,dy) = \omega^S_{\alpha}(s,B_1) \leq \omega^T_{\alpha}(s,B_1) = \mathbf{P}^s(T^{B_1} \subset \infty).\]

By a formula of S.Port [13],
\[C_\alpha(B_1) = \lim_{s \to \infty} A(n,\alpha)^{-1} |s|^{n-\alpha} \mathbf{P}^s(T^{B_1} \subset \infty).\]

Hence $\lim_{s \to \infty} H_{f_k}(s) = 0$. This implies that
\[\liminf_{S_+ \ni s \to \infty} v(s) = \liminf_{S_+ \ni s \to \infty} \varepsilon U_\lambda(s) \geq 0. \tag{3.3}\]
Note here that we cannot apply the minimum principle of subsection 2.6 because the condition \( v \geq 0 \) in \((S_+)^c\) is not satisfied. Nevertheless, we will prove that \( v \geq 0 \) in \( S_+ \). Suppose that \( v \) takes on strictly negative values in \( S_+ \). Let 

\[
\beta := \inf \{ v(s) : s \in S_+ \}.
\]

Take a sequence \( \{ s_k \} \) in \( S_+ \) such that \( v(s_k) \rightarrow \beta \). By passing to a subsequence if necessary, we may assume that \( \{ s_k \} \) converges in \( S_+ \). By the Cases 1-4 that we examined above, we may assume that \( \lim s_k = s_o \in S_+ \). The measure \( \lambda \) is not necessarily concentrated on \( E \) (see [12, p.181]). However, \( \lambda \) may be taken so that its support is as close to \( E \) as we wish; (see the proof of Theorem 3.1 in [12]). It is also known [12, Ch.I, §6] that the potential \( U^\lambda_o \) is an \( \alpha \)-harmonic function in the complement of the support of \( \lambda \). Hence \( v \) is \( \alpha \)-harmonic in a neighborhood of \( s_o \).

Hence

\[
0 = \Delta^{\alpha/2} v(s_o) = \int_{\mathbb{R}^n} \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} \, m_n(dx)
\]

\[
= \int_{R^n_+} \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} \, m_n(dx) + \int_{R^n_+} \frac{v(\hat{x}) - v(s_o)}{|\hat{x} - s_o|^{n+\alpha}} \, m_n(dx)
\]

(3.4)

\[
\geq \int_{R^n_+} \left[ \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} - \frac{v(x) + v(s_o)}{|x - s_o|^{n+\alpha}} \right] \, m_n(dx) =: I_1.
\]

We used above the equalities \( v(\hat{x}) = -v(x) + 2\varepsilon U^\lambda_o(x) \), \( U^\lambda_o(x) = U^\lambda_o(x) \), and \( |x - s_o| = |\hat{x} - s_o| \) which come from symmetry. Now we set \( A_1 = \{ x \in \mathbb{R}^n_+ : v(x) + v(s_o) \geq 0 \} \) and \( A_2 = \{ x \in \mathbb{R}^n_+ : v(x) + v(s_o) < 0 \} \). Using also the obvious inequality \( |x - s_o| > |x - s_o| \), we get

\[
I_1 = \int_{A_1} \left[ \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} - \frac{v(x) + v(s_o)}{|x - s_o|^{n+\alpha}} \right] \, m_n(dx)
\]

\[
+ \int_{A_2} \left[ \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} - \frac{v(x) + v(s_o)}{|x - s_o|^{n+\alpha}} \right] \, m_n(dx)
\]

\[
\geq \int_{A_1} \left[ \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} - \frac{v(x) + v(s_o)}{|x - s_o|^{n+\alpha}} \right] \, m_n(dx)
\]

\[
+ \int_{A_2} \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} \, m_n(dx)
\]

\[
= \int_{A_1} \frac{-2v(s_o)}{|x - s_o|^{n+\alpha}} \, m_n(dx) + \int_{A_2} \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} \, m_n(dx).
\]

Since \( v(s_o) < 0 \), the first integrand is positive. The second integrand is non-negative; indeed, if \( x \in \mathbb{R}^n_+ \setminus S_+ \), then \( v(x) - v(s_o) = f_k(x) + \varepsilon U^\lambda_o(x) - v(s_o) \geq 0 \) and if \( x \in S_+ \), then \( v(x) - v(s_o) \geq 0 \) by the definition of \( s_o \). Because of (3.4), we conclude that \( m_n(A_1) = 0 \) and \( v = v(s_o) \) a.e. in \( A_2 \). Hence \( v = v(s_o) < 0 \) a.e. in \( \mathbb{R}^n_+ \).

We proved above that the function \( v \) is equal to a negative constant a.e. in \( \mathbb{R}^n_+ \). This is absurd; indeed: (a) if \( m_n(\mathbb{R}^n_+ \setminus S_+) > 0 \) and \( x \in \mathbb{R}^n_+ \setminus S_+ \), then \( v(x) = f_k(x) + \varepsilon U^\lambda_o(x) \geq 0 \), (b) if \( m_n(\mathbb{R}^n_+ \setminus S_+ ) = 0 \), then \( S \) is unbounded and by (3.3), \( \lim_{S_+ \ni s \rightarrow -\infty} v(s) \geq 0 \).
The contradiction shows that \( v(s) \geq 0 \) for all \( s \in S_+ \). Since \( \varepsilon > 0 \) is arbitrary, (3.1) is proved. □

**Theorem 2.** Let \( D \) be an open set in \( \mathbb{R}^n \). Suppose that \( D \) is polarized with respect to the plane \( \Pi \). Let \( B \subset \mathbb{R}^n_+ \cap D^c \) be a Borel set. Then

(i) \( \omega^D_\alpha(x, B) \geq \omega^D_\alpha(\hat{x}, B) \), \( x \in \mathbb{R}^n_+ \cup \Pi \);

(ii) \( \omega^D_\alpha(x, B) \geq \omega^D_\alpha(x, \hat{B}) \), \( x \in \mathbb{R}^n_+ \cup \Pi \);

(iii) \( \omega^D_\alpha(x, B) + \omega^D_\alpha(\hat{x}, B) \geq \omega^D_\alpha(x, \hat{B}) + \omega^D_\alpha(\hat{x}, \hat{B}) \), \( x \in \mathbb{R}^n \);

(iv) \( \omega^D_\alpha(x, B) + \omega^D_\alpha(\hat{x}, B) \geq \omega^D_\alpha(\hat{x}, B) + \omega^D_\alpha(\hat{x}, \hat{B}) \), \( x \in \mathbb{R}^n \).

![Figure 2: An illustration for Theorem 2.](image-url)

**Proof.** Since \( D \) is polarized, the lower non-symmetric part of \( D \) is empty. Hence \( D = S \cup U \), where \( S \) is the symmetric part of \( D \) and \( U \) is the upper non-symmetric part of \( D \).

(i) If \( x \in (\mathbb{R}^n_+ \cup \Pi) \setminus S_+ \), the inequality (i) is trivial. So we assume that \( x = s \in S_+ \). By the strong Markov property,

\[
\omega^D_\alpha(s, B) = \omega^S_\alpha(s, B) + \int_U \omega^S_\alpha(s, du)\omega^D_\alpha(u, B)
\]

and

\[
\omega^D_\alpha(s, B) = \omega^S_\alpha(s, B) + \int_U \omega^S_\alpha(s, du)\omega^D_\alpha(u, B).
\]

By Theorem 1, \( \omega^S_\alpha(s, B) \geq \omega^S_\alpha(s, \hat{B}) \) and \( \omega^S_\alpha(s, du) \geq \omega^S_\alpha(s, du) \). So the inequality (i) is proved.

(ii) As in the proof of (i), we may assume that \( x = s \in S_+ \). Set \( S_1 := S \cup U \cup \hat{U} \). Then \( S_1 \) is an open set which is symmetric with respect to \( \Pi \) and contains \( D \). By the strong Markov property,

\[
\omega^D_\alpha(s, B) = \omega^{S_1}_\alpha(s, B) - \int_U \omega^D_\alpha(s, du)\omega^{S_1}_\alpha(u, B)
\]

and

\[
\omega^D_\alpha(s, \hat{B}) = \omega^{S_1}_\alpha(s, \hat{B}) - \int_U \omega^D_\alpha(s, du)\omega^{S_1}_\alpha(u, \hat{B}).
\]

By Theorem 1, \( \omega^{S_1}_\alpha(s, B) \geq \omega^{S_1}_\alpha(s, \hat{B}) \) and \( \omega^{S_1}_\alpha(u, \hat{B}) \geq \omega^{S_1}_\alpha(u, B) \), \( u \in \hat{U} \). So the inequality (ii) is proved.
(iii) By the inner regularity of $\alpha$-harmonic measure, we may and do assume that $B$ is a compact set in $\mathbb{R}^n \cap D^c$. Take a decreasing sequence of continuous functions $f_k : D^c \to [0,1]$ with $\text{supp} f_k \downarrow B$, $f_k \downarrow \chi_B$ and $f_k = 0$ in $(D^c)_{-}$. Let $f_k(x) = f_k(\hat{x})$, $x \in D^c$; ($f_k = 0$ in $\hat{U}$). Consider the sequences of functions

$$H_{f_k}(x) := \int_{D^c} f_k(y) \omega_\alpha^D(x,dy), \ x \in \mathbb{R}^n,$$

and

$$H_{\hat{f}_k}(x) := \int_{D^c} \hat{f}_k(y) \omega_\alpha^D(x,dy), \ x \in \mathbb{R}^n.$$&n

We have $H_{f_k}(x) \downarrow \omega_\alpha^D(x,B)$ and $H_{\hat{f}_k}(x) \downarrow \omega_\alpha^D(x,\hat{B})$, $x \in \mathbb{R}^n$. Therefore it suffices to prove that

$$H_{f_k}(x) + H_{f_k}(\hat{x}) \geq H_{\hat{f}_k}(x) + H_{\hat{f}_k}(\hat{x}), \ x \in \mathbb{R}^n, k \in \mathbb{N}.$$&n

Fix $k \in \mathbb{N}$ and define

$$v(x) = H_{f_k}(x) + H_{f_k}(\hat{x}) - H_{\hat{f}_k}(x) - H_{\hat{f}_k}(\hat{x}), \ x \in \mathbb{R}^n.$$&n

It is clear that $v$ is $\alpha$-harmonic in $S$. Note that for $u \in U$, $v(u) = H_{f_k}(u) - H_{\hat{f}_k}(u)$. So $v$ is $\alpha$-harmonic in $D$. It is also continuous in $\overline{D} \setminus E$, where $E$ is the set of irregular points of $\partial D$. We will apply the minimum principle (Lemma 5) to the function $v$ in the domain $D$.

Let $\zeta \in D^c$.

Case 1: If $\zeta \in \partial D \setminus (E \cup \hat{U})$, then

$$\lim_{D \ni x \to \zeta} v(x) = f_k(\zeta) + f_k(\hat{\zeta}) - \hat{f}_k(\zeta) - \hat{f}_k(\hat{\zeta}) = 0.$$&n

Case 2: If $\zeta \in (\partial D \cap \hat{U}) \setminus E$, then

$$\lim_{D \ni x \to \zeta} v(x) = f_k(\zeta) + H_{f_k}(\hat{\zeta}) - \hat{f}_k(\zeta) - H_{\hat{f}_k}(\hat{\zeta}) = H_{f_k}(\hat{\zeta}) - H_{\hat{f}_k}(\hat{\zeta})$$

$$= \int_{D^c} f_k(y) \omega_\alpha^D(\hat{\zeta},dy) - \int_{D^c} \hat{f}_k(y) \omega_\alpha^D(\hat{\zeta},dy)$$

$$= \int_{D^c} f_k(y) \omega_\alpha^D(\hat{\zeta},dy) - \int_{D^c} f_k(y) \omega_\alpha^D(\hat{\zeta},\overline{\partial}dy)$$

$$= \int_{(D^c)_{+}} f_k(y) [\omega_\alpha^D(\hat{\zeta},dy) - \omega_\alpha^D(\hat{\zeta},\overline{\partial}dy)] \geq 0.$$&n

Here $\omega_\alpha^D(\hat{\zeta},\overline{\partial}dy)$ is the measure $\mu$ on $(D^c)_{+}$ defined by $\mu(E) := \omega_\alpha^D(\hat{\zeta},E)$. The last equality holds because $f_k$ is supported in $(D^c)_{+}$. The inequality comes from part (ii) of Theorem 2.

Case 3: If $\zeta \in (D^c) \setminus \hat{U}$, then $v(\zeta) = f_k(\zeta) + f_k(\hat{\zeta}) - \hat{f}_k(\zeta) - \hat{f}_k(\hat{\zeta}) = 0$.

Case 4: If $x = u \in \hat{U} \setminus \partial D$, then we work as in Case 2.

By Lemma 5, we conclude that $v \geq 0$ on $D$.

(iv) The proof is similar to the proof of (iii). □
Theorem 3. Let $D$ be an open set in $\mathbb{R}^n$. Suppose that $D$ is polarized with respect to the plane $\Pi$. Let $B \subset \mathbb{R}^n \cap D^c$ be a Borel set. Then

(i) $\omega_D^\alpha(x, B) \leq \frac{1}{2}$, $x \in D_+ \cup D_0$;
(ii) $\omega_D^\alpha(x, \hat{B}) \leq \frac{1}{2}$, $x \in D_+ \cup D_0$;
(iii) $\omega_D^\alpha(x, B) \leq \frac{1}{2}$, $x \in (\hat{D})_- \cup D_0$.

Proof. We will prove only the inequality (ii). The proof of (i) is similar and (iii) is equivalent to (ii) because of symmetry.

We write $D = S \cup U$, where $S$ is the symmetric part of $D$ and $U$ is the upper non-symmetric part of $D$. Set $S_1 := D \cup \hat{U}$. Then $S_1$ is an open set, symmetric with respect to $\Pi$, and $D \subset S_1$. Using Theorem 1 we obtain

$$\omega_D^\alpha(x, \hat{B}) \leq \omega_{S_1}^\alpha(x, \hat{B}) \leq \omega_{S_1}^\alpha(x, B), \quad x \in D_+ \cup D_0.$$ 

Hence

$$\omega_D^\alpha(x, \hat{B}) \leq \frac{1}{2} \left[ \omega_{S_1}^\alpha(x, \hat{B}) + \omega_{S_1}^\alpha(x, B) \right] = \frac{1}{2} \omega_{S_1}^\alpha(x, B \cup \hat{B}) \leq \frac{1}{2}.$$

□

We now turn to a sharp form of Theorem 2.

Theorem 4. Let $D$ be an open set in $\mathbb{R}^n$. Suppose that $D$ is polarized with respect to the plane $\Pi$. Let $B \subset \mathbb{R}^n \cap D^c$ be a Borel set which is not $D$-null. Then for $x \in D_+$, we have

(3.5) $\omega_D^\alpha(x, B) > \omega_D^\alpha(\hat{x}, B)$

and

(3.6) $\omega_D^\alpha(x, B) > \omega_D^\alpha(x, \hat{B})$.

Proof. First we prove (3.5). We write $D = S \cup U$, where $S$ is the symmetric part of $D$ and $U$ is the upper non-symmetric part of $D$. If $x = u \in U$, then $\omega_D^\alpha(u, B) > 0$ because $B$ is not $D$-null. On the other hand, $\omega_D^\alpha(\hat{u}, B) = 0$ because $\hat{u} \notin B$. Therefore (3.5) is proved in this case. So it remains to prove (3.5) for $x = s \in S_+$. Consider the function

$$v(x) = \omega_D^\alpha(x, B) - \omega_D^\alpha(\hat{x}, B), \quad x \in \mathbb{R}^n.$$
Then \( v \) is \( \alpha \)-harmonic in \( D \) and by Theorem 2,
\[
(3.7) \quad v(x) \geq 0, \quad x \in \mathbb{R}^n_+.
\]
Also, it is obvious that
\[
(3.8) \quad v(x) + v(\hat{x}) = 0, \quad x \in \mathbb{R}^n_+.
\]
We want to prove that
\[
(3.9) \quad v(s) > 0, \quad s \in S_+.
\]
Suppose that \( v(s_o) = 0 \) for some \( s_o \in S_+ \). Since \( v \) is \( \alpha \)-harmonic in \( D \),
\[
0 = \Delta^{\alpha/2} v(s_o) = \int_{\mathbb{R}^n} \frac{v(x) - v(s_o)}{|x - s_o|^{n+\alpha}} m_n(dx) = \int_{\mathbb{R}^n} \frac{v(x)}{|x - s_o|^{n+\alpha}} m_n(dx)
\]
\[
= \int_{\mathbb{R}^n} \left[ \frac{v(x)}{|x - s_o|^{n+\alpha}} - \frac{v(s_o)}{|s_o|^{n+\alpha}} \right] m_n(dx)
\]
\[
= I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 := \int_{S_o} v(s) \left[ \frac{1}{|s - s_o|^{n+\alpha}} - \frac{1}{|s_o|^{n+\alpha}} \right] m_s(ds),
\]
\[
I_2 := \int_U v(u) \left[ \frac{1}{|u - s_o|^{n+\alpha}} - \frac{1}{|u - s_o|^{n+\alpha}} \right] m_u(du)
\]
\[
= \int_U \omega^D_\alpha(u, B) \left[ \frac{1}{|u - s_o|^{n+\alpha}} - \frac{1}{|u - s_o|^{n+\alpha}} \right] m_u(du),
\]
\[
I_3 := \int_B v(x) \left[ \frac{1}{|x - s_o|^{n+\alpha}} - \frac{1}{|x - s_o|^{n+\alpha}} \right] m_n(dx),
\]
\[
= \int_B \left[ \frac{1}{|x - s_o|^{n+\alpha}} - \frac{1}{|x - s_o|^{n+\alpha}} \right] m_n(dx),
\]
\[
I_4 := \int_{(D_+) \cap B} v(x) \left[ \frac{1}{|x - s_o|^{n+\alpha}} - \frac{1}{|x - s_o|^{n+\alpha}} \right] m_n(dx).
\]
Since \( v = 0 \) in \( (D_+) \cap B \), we have \( I_4 = 0 \). Because of the obvious inequality
\[
|x - s_o| < |x - \hat{s}_o|, \quad x \in \mathbb{R}^n_+,
\]
the integrands in \( I_1, I_2, I_3 \) are non-negative. Therefore \( I_1 = I_2 = I_3 = 0 \). We conclude that \( m_n(U) = 0 \), \( m_n(B) = 0 \) and \( v = 0 \) m.a.e. in \( S \). Since \( v \) is continuous in \( D \), we conclude that \( v = 0 \) in \( S \) which means that
\[
(3.10) \quad \omega^D_\alpha(s, B) = \omega^D_\alpha(\hat{s}, B), \quad s \in S.
\]
The fact that \( m_n(B) = 0 \) implies that (see [4], [15]) the set \( B \cap (\overline{D})^c \) is \( D \)-null; hence the set \( B \cap \partial D \) is not \( D \)-null. Thus, by [15, Lemma 1], we have
\[
\sup_{x \in D} \omega^D_\alpha(x, B) = 1.
\]
Take a sequence \( \{x_k\} \) in \( D \) such that
\[
(3.11) \quad \lim_{k \to \infty} \omega^D_\alpha(x_k, B) = 1.
\]
By Theorem 3, we may assume that $\{x_k\} \subset D_+$. Since $D_+$ is open set and $m_\alpha(U) = 0$, every neighborhood of $x_k$ contains a point $s_k \in S_+, \ k \in \mathbb{N}$. So, by the continuity of $\alpha$-harmonic measure in $D$, we can choose a sequence $s_k$ in $S_+$ such that

$$\lim_{k \to \infty} \omega_\alpha^D(s_k, B) = 1.$$  

Then, again by Theorem 3,

$$\limsup_{k \to \infty} \omega_\alpha^D(s_k, B) \leq \frac{1}{2}.$$  

This together with (3.12) contradict (3.10). So (3.9) is proved.

We now turn to the proof of (3.6). We consider the function

$$h(x) = \omega_\alpha^D(x, B) - \omega_\alpha^D(x, \hat{B}), \ x \in \mathbb{R}^n.$$  

We know from Theorem 2 that

$$h(x) \geq 0, \ h(x) + h(\hat{x}) \geq 0, \ x \in \mathbb{R}^n_+.$$  

We want to prove that

$$h(x) > 0, \ x \in D_+.$$  

Suppose that $h(x_0) = 0$ for some $x_0 \in D_+$. Since $h$ is $\alpha$-harmonic in $D$,

$$0 = \Delta^{\alpha/2} h(x_0) = \int_{\mathbb{R}^n} h(x) - h(x_0) \frac{m_\alpha(dx)}{|x - x_0|^{n+\alpha}}$$

$$= \int_{\mathbb{R}^n_+} \frac{h(x)}{|x - x_0|^{n+\alpha}} m_\alpha(dx) + \int_{\mathbb{R}^n_+} \frac{h(x)}{|x - x_0|^{n+\alpha}} m_\alpha(dx)$$

$$= \int_{\mathbb{R}^n_+} \frac{h(x)}{|x - x_0|^{n+\alpha}} m_\alpha(dx) + \int_{\mathbb{R}^n_+} \frac{h(\hat{x})}{|\hat{x} - x_0|^{n+\alpha}} m_\alpha(dx)$$

$$= \int_{\mathbb{R}^n_+} \left\{ h(x) + h(\hat{x}) + h(x) \left[ \frac{1}{|x - x_0|^{n+\alpha}} - \frac{1}{|\hat{x} - x_0|^{n+\alpha}} \right] \right\} m_\alpha(dx) =: J.$$  

As in the proof of (3.5), we find that $J = J_1 + J_2 + J_3$, where

$$J_1 := \int_{S_+} \left\{ \frac{h(s) + h(\hat{s})}{|s - x_0|^{n+\alpha}} + h(s) \left[ \frac{1}{|s - x_0|^{n+\alpha}} - \frac{1}{|\hat{s} - x_0|^{n+\alpha}} \right] \right\} m_\alpha(ds),$$

$$J_2 := \int_{U} \frac{h(u)}{|u - x_0|^{n+\alpha}} m_\alpha(du),$$

$$J_3 := \int_{B} \left[ \frac{1}{|x - x_0|^{n+\alpha}} - \frac{1}{|x - x_0|^{n+\alpha}} \right] m_\alpha(dx).$$

Using (3.13) we conclude that $m_\alpha(B) = 0$ and that $v = 0$ in $S_+$ which means that

$$\omega_\alpha^D(s, B) = \omega_\alpha^D(s, \hat{B}), \ s \in S.$$  

By (3.15) and the fact that $B$ is not $D$-null we infer that $\hat{B}$ is not $D$-null. Since $m_\alpha(\hat{B}) = m_\alpha(B) = 0$, the set $\hat{B} \cap \partial D$ is not $D$-null. By [15, Lemma 1], we thus have

$$\sup_{x \in D} \omega_\alpha^D(x, \hat{B}) = 1.$$
Take a sequence \( \{y_k\} \) in \( D \) with \( \omega_\alpha^D(y_k, \hat{B}) \to 1 \). Since \( \hat{B} \subset \mathbb{R}^n \), Theorem 3 implies that we may assume that \( y_k \in D_+ = S_+, k \in \mathbb{N} \). Then (3.15) gives \( \omega_\alpha^D(y_k, B) \to 1 \). But Theorem 3 implies \( \omega_\alpha^D(y_k, B) \leq 1/2 \). This contradiction proves (3.14). \( \square \)

**Theorem 5.** Let \( D \) be an open set in \( \mathbb{R}^n \). Suppose that \( D \) is polarized with respect to the hyperplane \( \Pi \), i.e. \( D = S \cup U \), where \( S \) is the symmetric part of \( D \) and \( U \) is the upper non-symmetric part of \( D \). Let \( B \subset \mathbb{R}^n \cap D^c \) be a Borel set which is not \( D \)-null.

(i) \[
(3.16) \quad \omega^D_\alpha(s_\alpha, B) + \omega^D_\alpha(s_\alpha, \hat{B}) = \omega^D_\alpha(s_\alpha, \hat{B}) + \omega^D_\alpha(s_\alpha, \hat{B})
\]
for some \( s_\alpha \in S \) if and only if \( C_\alpha(U) = 0 \).

(ii) \[
(3.17) \quad \omega^D_\alpha(s_\alpha, B) + \omega^D_\alpha(s_\alpha, \hat{B}) = \omega^D_\alpha(s_\alpha, B) + \omega^D_\alpha(s_\alpha, \hat{B})
\]
for some \( s_\alpha \in S \) if and only if \( C_\alpha(U) = 0 \).

(iii) \[
(3.18) \quad \omega^D_\alpha(s_\alpha, B) = \omega^D_\alpha(s_\alpha, \hat{B})
\]
for some \( s_\alpha \in S_\alpha := S \cap \Pi \) if and only if \( C_\alpha(U) = 0 \).

**Proof.** We prove only (i); the proofs of (ii) and (iii) are similar.

Suppose that (3.16) holds for some \( s_\alpha \in S \). By the strong Markov property,

\[
(3.19) \quad \omega^D_\alpha(s_\alpha, B) = \omega^S_\alpha(s_\alpha, B) + \int_U \omega^S_\alpha(s_\alpha, du) \omega^D_\alpha(u, B),
\]

\[
(3.20) \quad \omega^D_\alpha(s_\alpha, \hat{B}) = \omega^S_\alpha(s_\alpha, \hat{B}) + \int_U \omega^S_\alpha(s_\alpha, du) \omega^D_\alpha(u, \hat{B}),
\]

\[
(3.21) \quad \omega^D_\alpha(s_\alpha, \hat{B}) = \omega^S_\alpha(s_\alpha, \hat{B}) + \int_U \omega^S_\alpha(s_\alpha, du) \omega^D_\alpha(u, \hat{B}),
\]

\[
(3.22) \quad \omega^D_\alpha(s_\alpha, \hat{B}) = \omega^S_\alpha(s_\alpha, \hat{B}) + \int_U \omega^S_\alpha(s_\alpha, du) \omega^D_\alpha(u, \hat{B}).
\]

Hence

\[
\int_U [\omega^S_\alpha(s_\alpha, du) + \omega^S_\alpha(s_\alpha, du)] \omega^D_\alpha(u, B) = \int_U [\omega^S_\alpha(s_\alpha, du) + \omega^S_\alpha(s_\alpha, du)] \omega^D_\alpha(u, \hat{B}).
\]

By Theorem 4, \( \omega^S_\alpha(u, B) > \omega^D_\alpha(u, \hat{B}) \) for all \( u \in U \). Hence \( \omega^S_\alpha(s_\alpha, du) + \omega^S_\alpha(s_\alpha, du) \) is the zero measure on \( U \). This implies \( \omega^S_\alpha(s_\alpha, U) = 0 \), i.e. \( U \) is \( S \)-null. By Lemma 3, \( C_\alpha(U) = 0 \).

Conversely, if \( C_\alpha(U) = 0 \), then \( U \) is \( S \)-null. Therefore (3.19)-(3.22) imply

\[
\omega^D_\alpha(s, B) + \omega^D_\alpha(\hat{s}, B) = \omega^D_\alpha(s, \hat{B}) + \omega^D_\alpha(\hat{s}, \hat{B})
\]
for all \( s \in S \). \( \square \)

**Theorem 6.** Let \( D \) be an open set in \( \mathbb{R}^n \). Suppose that \( D \) is polarized with respect to the hyperplane \( \Pi \), i.e. \( D = S \cup U \), where \( S \) is the symmetric part of \( D \) and \( U \) is the upper non-symmetric part of \( D \). Let \( B \subset D^c \) be a Borel set which is symmetric with respect to \( \Pi \) and is not \( D \)-null. Then

\[
\omega^D_\alpha(s, B) = \omega^D_\alpha(\hat{s}, B)
\]
for some \( s \in S_+ \) if and only if \( C_\alpha(U) = 0 \).
Proof. Similar to the proof of Theorem 5.

References